IMPROVED BOUND ON THE MAXIMUM NUMBER OF CLIQUE-FREE COLORINGS WITH TWO AND THREE COLORS

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Abstract. Given integers $r, k \geq 2$ let $\kappa_{r,k+1}(G)$ denote the number of distinct edge colorings of G with r colors, which are K_{k+1}-free, i.e., which contain no monochromatic clique on $k+1$ vertices. Alon, Balogh, Keevash and Sudakov [*J. Lond. Math. Soc. 70 (2004)*] show that for $r \in \{2, 3\}$ and all $k \geq 2$ the maximum of $\kappa_{r,k+1}(G)$ over all G on n vertices is achieved only by the Turán graph, provided $n > n_0(k)$ is sufficiently large. The proof uses Szemerédi's regularity lemma and yields an $n_0(k)$ which is tower-type with height exponential in k. As a lower bound the authors observed that $n_0(k)$ must be at least exponential in k.

In this paper we essentially close the gap between the upper and the lower bound for $n_0(k)$. Answering the question posed by Alon et al. we show that the lower bound is of correct order and that it suffices to choose $n_0(k) = \exp(Ck^4)$ for some absolute constant C.

1. Introduction

Given a graph G and integers $r, k \geq 2$. By $\kappa_{r,k+1}(G)$ we denote the number of edge colorings of G with r colors, which are K_{k+1}-*free*, i.e., which contain no monochromatic copy of the clique on $k+1$ vertices. We are interested in $\kappa_{r,k+1}(n)$, the maximum value of $\kappa_{r,k+1}(G)$ over all graphs G on n vertices, and in the graphs G which attain this value.

A straightforward lower bound for $\kappa_{r,k+1}(n)$ is obtained from the maximum K_{k+1}-free graph on n vertices, determined by Turán's theorem [12] to be the complete k-partite graph with partition classes as equal as possible. This commonly called *Turán graph* we denote by $T_k(n)$ and by $t_k(n)$ we denote its number of edges. Then $\kappa_{r,k+1}(n) \geq \kappa_{r,k+1}\big(T_k(n)\big) = r^{t_k(n)}$ and it was conjectured by Erdős and Rothschild [4, 5] that this bound is sharp for $r = 2$ and $k = 2$, i.e., $\kappa_{2,3}(n) = 2^{t_2(n)}$, provided n is sufficiently large. The conjecture was confirmed by Yuster in [13] and subsequently, Alon, Balogh, Keevash and Sudakov [1] extend the result to $r \in \{2, 3\}$ and all $k \geq 2$, again provided n is sufficiently large.

This phenomenon does not persist for $r \geq 4$ and the only exact results for this range are due to Pikhurko and Yilma [10] who, building on the work of Alon et al., determine the unique maximizers for $(r, k) = (4, 2)$ and for $(r, k) = (4, 3)$. These turn out to be the $T_4(n)$ for $(r, k) = (4, 2)$ and $T_9(n)$ for $(r, k) = (4, 3)$ (see also [9] for some recent progress).

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In general, however, the problem appears to be very difficult for $r \geq 4$. Thus, the question is only well understood for all $k \geq 2$ in case $r = 2$ and $r = 3$.

Theorem 1 ([1, Theorem 1.1]). *Let $r \in \{2, 3\}$. For every integer $k \geq 2$ there is an $n_0 = n_0(k)$ such that the following holds. If G is a graph on n vertices such that the number of K_{k+1}-free r-colorings of G is at least $r^{t_k(n)}$, then G is isomorphic to the Turán graph $T_k(n)$.*

The proof of Theorem 1 uses Szemerédi's regularity lemma with regularity parameter 10^{-8k} and yields therefore an n_0 of the form $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$, a tower of 2 with height exponential in k (see [6]). As a lower bound the authors observe that n_0 must be exponential in k for the result to hold. Indeed, for $n < r^{\frac{k-1}{2}}$ a random r-coloring of K_n contains no monochromatic K_{k+1} with probability larger than 1/2, showing that $\kappa_{r,k+1}(n) > \frac{1}{2} r^{\binom{n}{2}}$ for these values of n. Alon et al. put forth the question to narrow the gap between the upper and the lower bound on $n_0 = n(r, k)$.

In this paper we address this question and show that the lower bound is of correct order. Our result states that it is sufficient to choose $n_0 = \exp(Ck^4)$ for some absolute constant C.

Theorem 2. *There exists an absolute constant $c \in (0, 1)$ such that the following holds. Let $r \in \{2, 3\}$ and $2 \leq k = k(n) \leq c(\log n)^{1/4}$. If G is a graph on n vertices such that the number of K_{k+1}-free r-colorings of G is at least $r^{t_k(n)}$, then G is isomorphic to the Turán graph $T_k(n)$.*

Similar to the proof of Theorem 1 in [1] our proof of Theorem 2 splits into two parts, a stability part and an exact part. While proof of the exact part in [1] only requires an exponential dependency between n_0 and k, it is the proof of stability which uses the regularity lemma.

Our main technical lemma, Lemma 3, improves upon this. A graph G is called *t-close to being k-partite* if there is a partition $V(G) = V_1 \cup \cdots \cup V_k$ such that $\sum_{i \in [k]} e(V_i) \leq t$.

Lemma 3. *Given $r \in \{2, 3\}$. There exists an absolute constant $c_0 \in (0, 1)$ such that the following holds for $3 \leq k = k(n) \leq c_0(\log n)^{1/4}$. Suppose G is a graph on n vertices with $\log_r \kappa_{r,k+1}(G) \geq t_k(n) - n^{2-1/k^2}$. Then G is $250n^{2-1/k^2}$-close to being k-partite.*

The proof of Lemma 3, given in the next section, relies on the container method developed by Balogh, Morris and Samotij [3] and, independently, by Saxton and Thomason [11]. The exact part then simply follows from the result of Alon et al., which states that if stability as above holds for all graphs on $n_1 \geq n_0$ vertices, for some $n_0 = n_0(r, k)$, then the exact result holds for all graphs on $n \geq n_0^2$ vertices.

Lemma 4 ([1, Proof of Theorem 1.1]). *Let $r \in \{2, 3\}$ and $k \geq 2$. Suppose $n_0 = n_0(k)$ satisfies the property that every graph G_1 on $n_1 \geq n_0$ vertices with $\kappa_{r,k+1}(G_1) \geq r^{t_k(n_1)}$ is $10^{-8k}{n_1}^2$-close to being k-partite. Then each graph G on $n \geq n_0^2$ vertices with $\kappa_{r,k+1}(G) \geq r^{t_k(n)}$ is isomorphic to the Turán graph $T_k(n)$.* □

We note that Lemma 4 is not explicitly stated in [1], however, the proof of Theorem 1 based on their version of stability yields exactly this. Given Lemma 3 and Lemma 4, Theorem 2 immediately follows.

Proof of Theorem 2. Let c_0 denote the absolute constant from Lemma 3, let $c_0' = \min\{c_0, 1/2\}$ and let $c = c_0'/2$. Given $r \in \{2,3\}$ and $k \geq 2$. Define $n_0 = n_0(k) = \exp\left(\frac{k}{c_0'}\right)^4$. Lemma 3 then implies that any graph G_1 on $n_1 \geq n_0$ vertices with $\kappa_{r,k+1}(G_1) \geq r^{t_k(n_1)}$ is $\left(250 n_1^{2-1/k^2}\right)$-close to being k-partite. By the choice of n_0 we have $250{n_1}^{2-1/k^2} \leq 10^{-8k}{n_1}^2$ for every $n_1 \geq n_0$. Hence, the hypothesis of Lemma 4 holds for the chosen n_0. This lemma then implies that any G on $n \geq \exp\left(\frac{k}{c}\right)^4 \geq n_0^2$ vertices with $\kappa_{r,k+1}(G) \geq r^{t_k(n)}$ is isomorphic to the Turán graph $T_k(n)$. $\square$

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2. Proof of Lemma 3

As mentioned above the proof of Lemma 3 relies on an application of the container method developed by Balogh, Morris and Samotij [3] and, independently, by Saxton and Thomason [11]. More generally, the proof shows that this method may replace regularity lemmas in the study of Erdős-Rothschild type problems. Indeed, in a parallel work [7] we apply the container method to study sum-free colorings of subsets of abelian groups.

For our current problem the container type result we shall use is Theorem 5 from below. Given any graph G, the theorem roughly states that *all* K_{k+1}-free subgraphs of G can be *captured* by a *small* number of almost K_{k+1}-free subgraphs of G. Theorem 5 is a slight modification of Theorem 3.2 from [2], which in turn follows from an application of the results in [11] as done in [8], for example. By $K_{k+1}(C)$ we denote the number of copies of K_{k+1} in the graph C.

Theorem 5 ([2, Theorem 3.2])**.** *There exists an absolute constant $c \in (0,1)$ such that the following holds. Let $k = k(n) \in \mathbb{N}$ be a function satisfying $k \leq c(\log n)^{1/4}$ and let G be a graph on n vertices. Then there exists a collection $\mathcal{C} = \mathcal{C}(G)$ of graphs, each on n vertices, such that the following holds:*

(a) every K_{k+1}-free subgraph $G' \subset G$ is a subgraph of some $C \in \mathcal{C}$,
(b) $K_{k+1}(C) \leq n^{k+1-2/k^2}$ for every $C \in \mathcal{C}$,
(c) $|\mathcal{C}| \leq \exp\left(n^{2-2/k^2}\right)$. $\square$

Theorem 3.2 from [2] is stated for $G = K_n$ only, however, by taking the intersection of each container C with G we obviously obtain the family $\mathcal{C} = \mathcal{C}(G)$ from above.

Further, we require the following result from [2].

Theorem 6 ([2, Theorem 1.2])**.** *For every positive integers n, t, k the following holds. Every graph G which is not t-close to being k-partite contains at least*

$$\frac{n^{k-1}}{e^{2k} \cdot k!}\left(e(G) + t - \left(1 - \frac{1}{k}\right)\frac{n^2}{2}\right)$$

copies of K_{k+1}. $\square$

We are now in the position to give the proof of Lemma 3.

Proof of Lemma 3. We give the proof for $r = 3$ only. The proof for $r = 2$ follows the same line. Let $c = c_0/2$, where c_0 is the absolute constant obtained from Theorem 5. Given G as stated in Lemma 3 and let $\mathcal{C} = \mathcal{C}(G)$ be a container family as in Theorem 5. Let $\Phi(G) = \Phi_{k+1}(G)$ denote the set of all K_{k+1}-free 3-colorings of G. For each $\varphi \in \Phi(G)$ note that $\big(\varphi^{-1}(1), \varphi^{-1}(2), \varphi^{-1}(3)\big)$ is a triple of K_{k+1}-free subgraphs of G which, moreover, cover all edges of G. Due to property (a) of Theorem 5 we can therefore assign to each $\varphi \in \Phi$ a triple $(C_1, C_2, C_3) \in \mathcal{C}^3$ such that $\varphi^{-1}(i) \subseteq C_i$ for every $i \in [3]$. Let $\Phi(C_1, C_2, C_3)$ denote the set of all $\varphi \in \Phi(G)$ assigned to (C_1, C_2, C_3) and let $(H_1, H_2, H_3) \in \mathcal{C}^3$ be a triple which maximizes $|\Phi(C_1, C_2, C_3)|$, where the maximum is taken over all $(C_1, C_2, C_3) \in \mathcal{C}^3$. Owing to (c) of Theorem 5 we have

$$|\Phi(G)| \leq \sum_{(C_1,C_2,C_3)\in\mathcal{C}^3} |\Phi(C_1, C_2, C_3)| \leq |\mathcal{C}|^3 \cdot |\Phi(H_1, H_2, H_3)| \leq e^{3n^{2-2/k^2}} \cdot |\Phi(H_1, H_2, H_3)|.$$

Combining with the lower bound $\log_3 |\Phi(G)| \geq t_k(n) - n^{2-1/k^2}$ this yields

$$\log_3 |\Phi(H_1, H_2, H_3)| > t_k(n) - 2n^{2-1/k^2}. \tag{1}$$

Suppose an edge $e \in G$ is not contained in H_j for some $j \in [3]$. Then for any $\varphi \in \Phi(H_1, H_2, H_3)$ we have $\varphi(e) \neq j$ since $\varphi^{-1}(i) \subseteq H_i$ for all $i \in [3]$. In particular, if $e \in G$ is contained in ℓ members of (H_1, H_2, H_3) then there are at most ℓ choices of colors for e, i.e. $\big|\big\{\varphi(e) \colon \varphi \in \Phi(H_1, H_2, H_3)\big\}\big| \leq \ell$. With this in mind let m_ℓ be the number of edges in $G = H_1 \cup H_2 \cup H_3$ which are contained in exactly ℓ members of (H_1, H_2, H_3). Then

$$|\Phi(H_1, H_2, H_3)| \leq 2^{m_2} 3^{m_3} \qquad \text{and} \qquad m_1 + 2m_2 + 3m_3 = |H_1| + |H_2| + |H_3|. \tag{2}$$

To obtain an upper bound on m_2 we argue that $|H_i| \leq t_k(n) + n^{2-1/k^2}$ for each $i \in [3]$. Indeed, if the opposite holds then some H_i is not $\left(n^{2-1/k^2}\right)$-close to being k-partite, as the Turán graph $T_k(n)$ maximizes the number of edges among all k-partite graphs on n vertices. Then Theorem 6, together with $t_k(n) > \left(1 - \frac{1}{k}\right)\frac{n^2}{2} - kn$ and $k < \frac{1}{2}(\log n)^{1/4}$, would imply

$$K_{k+1}(H_i) \geq \frac{n^{k-1}}{e^{2k}k!} n^{2-1/k^2} > \frac{n^{k+1-1/k^2}}{e^{4k\log k}} > n^{k+1-2/k^2}$$

which contradicts (b) of Theorem 5. Hence, $|H_1| + |H_2| + |H_3| \leq 3\left(t_k(n) + n^{2-1/k^2}\right)$ and the second part of (2) yields

$$m_2 \leq \frac{3}{2}\left(t_k(n) + n^{2-1/k^2} - m_3\right). \tag{3}$$

Altogether we obtain

$$t_k(n) - 2n^{2-\frac{1}{k^2}} \overset{(1)}{\leq} \log_3 |\Phi(H_1, H_2, H_3)| \overset{(2)}{\leq} m_3 + m_2 \log_3 2 \overset{(3)}{\leq} m_3 + \frac{3}{2}\log_3 2\left(t_k(n) + n^{2-\frac{1}{k^2}} - m_3\right)$$

which yields $m_3 \geq t_k(n) - 60n^{2-1/k^2}$ after rearranging and using $\frac{3}{2}\log_3 2 < \frac{19}{20} < 1$.

Let $H = H_1 \cap H_2 \cap H_3$ which is a graph on n vertices with $m_3 \geq t_k(n) - 60n^{2-1/k^2}$ edges and $K_{k+1}(H) \leq n^{k+1-2/k^2}$. Thus H is $\left(65n^{2-1/k^2}\right)$-close to being k-partite, as otherwise Theorem 6 would imply a contradiction on $K_{k+1}(H)$ by a similar calculation as above. As $G = H_1 \cup H_2 \cup H_3$

and $|H_i \setminus H| = |H_i| - |H| \leq (t_k(n) + n^{2-1/k^2}) - (t_k(n) - 60n^{2-1/k^2}) = 61n^{2-1/k^2}$ for $i = 1, 2, 3$, we have $|G \setminus H| \leq 183n^{2-1/k^2}$. Thus G is $\left(250n^{2-1/k^2}\right)$-close to being k-partite, as required. □

References

[1] N. Alon, J. Balogh, P. Keevash, and B. Sudakov. The number of edge colorings with no monochromatic cliques. *J. Lond. Math. Soc., II. Ser.*, 70(2):273–288, 2004.

[2] J. Balogh, N. Bushaw, M. Collares, H. Liu, R. Morris, and M. Sharifzadeh. The typical structure of graphs with no large cliques. *Combinatorica*, 37(4):617–632, 2017.

[3] J. Balogh, R. Morris, and W. Samotij. Independent sets in hypergraphs. *J. Am. Math. Soc.*, 28(3):669–709, 2015.

[4] P. Erdős. Old and new problems in combinatorial analysis and graph theory. In *Second International Conference on Combinatorial Mathematics (New York, 1978)*, volume 319 of *Ann. New York Acad. Sci.*, pages 177–187. New York Acad. Sci., New York, 1979.

[5] P. Erdős. Some of my favourite problems in various branches of combinatorics. *Matematiche (Catania)*, 47(2):231–240 (1993), 1992. Combinatorics 92 (Catania, 1992).

[6] W.T. Gowers. Lower bounds of tower type for szemerédi's uniformity lemma. *Geometric & Functional Analysis GAFA*, 7(2):322–337, 1997.

[7] H. Hàn and A. Jiménez. Maximum number of sum-free colorings in finite abelian groups. *Israel Journal of Mathematics*, accepted.

[8] F. Mousset, R. Nenadov, and A. Steger. On the number of graphs without large cliques. *SIAM Journal on Discrete Mathematics*, 28(4):1980–1986, 2014.

[9] O. Pikhurko, K. Staden, and Z. B. Yilma. The Erdős-Rothschild problem on edge-colourings with forbidden monochromatic cliques. *Math. Proc Cambridge Phil. Soc.*, to appear.

[10] O. Pikhurko and Z. B. Yilma. The maximum number of K_3-free and K_4-free edge 4-colorings. *J. Lond. Math. Soc., II. Ser.*, 85(3):593–615, 2012.

[11] D. Saxton and A. Thomason. Hypergraph containers. *Invent. Math.*, 201(3):925–992, 2015.

[12] P. Turán. Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok*, 48:436–452, 1941.

[13] R. Yuster. The number of edge colorings with no monochromatic triangle. *J. Graph Theory*, 21(4):441–452, 1996.

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