Deciding whether a Grid is a Topological Subgraph of a Planar Graph is NP-Complete

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Abstract

The Topological Subgraph Containment (TSC) Problem is to decide, for two given graphs G and H, whether H is a topological subgraph of G. It is known that the TSC Problem is NP-complete when H is part of the input, that it can be solved in polynomial time when H is fixed, and that it is fixed-parameter tractable by the order of H.

Motivated by the great significance of grids in graph theory and algorithms due to the Grid-Minor Theorem by Robertson and Seymour, we investigate the computational complexity of the Grid TSC Problem in planar graphs. More precisely, we study the following decision problem: given a positive integer k and a planar graph G, is the $k \times k$ grid a topological subgraph of G? We prove that this problem is NP-complete, even when restricted to planar graphs of maximum degree six, via a novel reduction from the Planar Monotone 3-SAT Problem.

Keywords: topological subgraph, subgraph homeomorphism, subdivision, grids, planar graph, NP-complete

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1. Introduction

Given a graph G, the *subdivision* of an edge uv of G consists of its deletion and the addition of a new path of length two with ends u and v. For two graphs G and H, we say that H is a *topological subgraph* of G, or that G *contains a subdivision* of H, if G has a subgraph isomorphic to a graph obtained from H by repeatedly subdividing edges. This notion appears for example in the classical characterization of planar graphs by Kuratowski.

Our work concerns the computational complexity of the Grid Topological Subgraph Containment (Grid TSC) Problem in planar graphs. The general question, known as the Topological Subgraph Containment (TSC) Problem or as the Subgraph Homeomorphism Problem, is to determine for two given graphs G and H, whether H is a topological subgraph of G. To the best of our knowledge, investigations on this problem started with the work of LaPaugh and Rivest [9], who observed that, when H is part of the input, the TSC Problem is NP-complete. Indeed, when G is a graph on n vertices and H is a cycle on n vertices, then solving the TSC Problem means to decide whether G contains a Hamilton Cycle. Since the Hamilton Cycle Problem remains NP-complete when restricted to planar graphs [6], it also follows that the TSC Problem restricted to planar graphs is NP-complete. This is different when the graph H is fixed. An algorithmic result within the famous Graph Minor Theorem of Robertson and Seymour [10] is that, when H is fixed, the TSC Problem can be solved in time polynomial in the order of the input graph G. However, the constants involved in their result are enormous, which implies that their algorithm for the Topological Subgraph Problem is not practical. More efficient algorithms are known for certain graphs H, including the complete bipartite graph $K_{3,3}$ [1] and wheels with up to seven spokes [5, 13, 14]. More recently, Grohe et al. [7] showed that the TSC Problem is fixed-parameter tractable by the order of H. Their algorithm solves the TSC Problem in time proportional to $f(n_H) \cdot n_G^3$, where n_G and n_H denote the number of vertices of the given graphs G and H, respectively, and $f(n_H)$

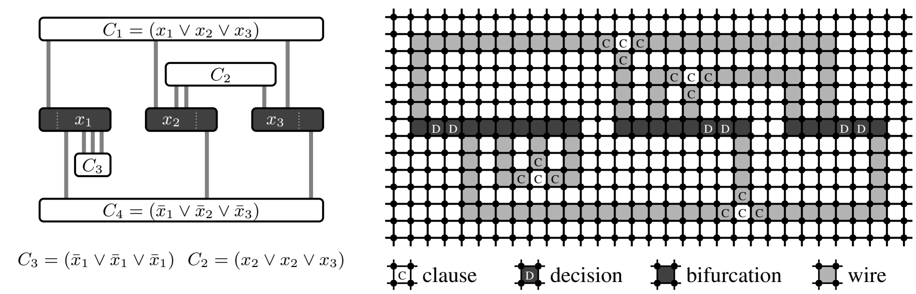
does not depend on n_G.

In this work, we prove that the TSC PROBLEM remains NP-complete when G is a planar graph and H is a grid. More precisely, we study the GRID TSC PROBLEM, which is to decide whether a given graph G contains the $k \times k$ grid as a topological subgraph, where k is part of the input. We show the following.

Theorem 1. *The* GRID TSC PROBLEM *in planar graphs is NP-complete, even when restricted to planar graphs with maximum degree 6.*

In other words, Theorem 1 says that finding largest topological grid minors in planar graphs with maximum degree 6 is NP-hard. Our proof of the previous theorem is a novel reduction, sketched in Section 1.1, from the PLANAR MONOTONE 3-SAT PROBLEM, which is NP-complete [2].

A concept related to topological subgraphs is the concept of minors. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from G by a series of edge contractions, and vertex and edge deletions. If G contains H as a topological subgraph, then G also contains H as a minor, and the reverse holds when $\Delta(H) \leq 3$. Moreover, for every graph H, there is a finite list of graphs $H_1, \dots, H_\ell$ such that G contains H as a minor if and only if G contains one of the graphs $H_1, \dots, H_\ell$ as a topological subgraph.
Grid minors are fundamental structures in graph theory and algorithms due to the Grid-Minor Theorem by Robertson and Seymour [12], which roughly states that graphs of large tree-width necessarily contain large grid minors. A recent improvement on the bound provided in [12] can be found in [3] and tighter bounds for grid minors in planar graphs are presented in [11]. Since many NP-hard problems can be solved efficiently when restricted to graphs of bounded tree-width, the problem of finding largest grid minors is of great interest. Currently, the best approximation for finding a largest grid minor in a planar graph is due to Gu and Tamaki [8]. The previous problem is NP-hard for general graphs and it is open whether it remains NP-hard when restricted to planar graphs. Due to the close relation between topological grid subgraphs and grid minors, we believe that our work also provides important insights into



a) A monotone rectilinear representation of $\phi = \bigwedge C_i$.

b) Plan for adding gadgets to the grid $\tilde{G}$.

Figure 1

the resolution of the computational complexity of the problem of finding largest grid minors in planar graphs.

1.1. Reduction Idea

Consider an instance of the Planar Monotone 3-SAT Problem, i.e., a drawing $\mathcal{R}$ as in Figure 1a). A formal definition is presented in Section 2.1. Denote by $\phi = \phi(\mathcal{U}, \mathcal{C})$ the 3-SAT formula corresponding to $\mathcal{R}$. To prove that the Grid TSC Problem in planar graphs is NP-hard (Theorem 1), we construct a planar graph G_ϕ and define a value k such that the $k \times k$ grid is a topological subgraph of G_ϕ if and only if ϕ is satisfiable.

The construction of G_ϕ starts with the $k \times k$ grid $\tilde{G}$. We present gadgets for clauses and variables as well as wire gadgets that are used to connect variable gadgets to clause gadgets. Each variable gadget consists of one decision gadget and several bifurcation gadgets. The purpose of the decision gadget is to encode whether a variable of ϕ is set to True or False and the purpose of the clause gadget is to ensure that at most two of the three literals of the clause are set to False. The bifurcation and wire gadgets duplicate and propagate the information through the graph G_ϕ. Using the drawing $\mathcal{R}$, these gadgets are placed into some *inner* faces of $\tilde{G}$, i.e., faces of $\tilde{G}$ that are bounded by a cycle of length 4, see Figure 1.

We show that, if G_ϕ contains a subdivision H of the $k \times k$ grid, then, roughly speaking, H is $\tilde{G}$ except for a few local differences. This is made more precise later on by introducing the concept of a normal subdivision. In the construction of G_ϕ, for each variable gadget, an edge of $\tilde{G}$ is deleted. Such a deletion forces the corresponding path of H to bend and, as a result, all paths along some wire gadgets connecting that variable to positive or negative clauses have to bend. Here, a bend of a path can be interpreted as a variable sending the value False towards a clause. The clause gadget, placed into an inner face f of $\tilde{G}$, is designed in such a way that at most two paths can bend into the face f. As a consequence, if the three variables connected to the clause gadget in f are set to False, one of the paths cannot bend into the face f and G_ϕ does not contain a subdivision of the $k \times k$ grid.

1.2. Organization

In Section 2, we introduce the Planar Monotone 3-SAT Problem as well as the terminology used for subdivisions of grids. In particular, we formally define normal subdivisions and bends. Afterward, in Section 3, we describe the gadgets in detail and state their key properties. In Section 4, these gadgets are used to construct a graph G_ϕ, which depends on a monotone rectilinear drawing of a 3-SAT formula ϕ. Finally, Section 5 presents the reduction, i.e., we prove that G_ϕ contains a subdivision of a grid of certain size if and only if ϕ is satisfiable.

2. Preliminaries

2.1. The Planar Monotone 3-SAT Problem

Consider a set $\mathcal{U} = \{x_1, \dots, x_n\}$ of boolean variables together with a set $\mathcal{C} = \{C_1, \dots, C_m\}$ of clauses over $\mathcal{U}$, where each clause C_i with $i \in \{1, \dots, m\}$ is a disjunction of at most 3 literals, i.e., variables from $\mathcal{U}$ or their negation. Then, $\phi = \phi(\mathcal{U}, \mathcal{C}) = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ is a *3-SAT formula* over $\mathcal{U}$ and is called *satisfiable* if there exists an assignment of True and False to the variables in $\mathcal{U}$ such that ϕ evaluates to TRUE. If a clause C contains only positive or only negative literals, then C is called *positive* or *negative*, respectively. A 3-SAT formula ϕ is called *monotone* if each clause in ϕ is positive or negative. Moreover, ϕ is called *planar* if the following bipartite graph G is planar: the vertex set of G is $\{x_1, \ldots, x_n\} \cup \{C_1, \ldots, C_m\}$ and $\{x, C\}$ is an edge of G if and only if the clause C uses x or its negation $\bar{x}$.

In [2], de Berg and Khosravi introduce monotone rectilinear representations, which combine the properties monotone and planar. Assume that $\phi = \phi(\mathcal{U}, \mathcal{C})$ is a monotone and planar 3-SAT formula. Consider an orthogonal coordinate system in the plane consisting of a horizontal and a vertical axis. A *monotone rectilinear representation* of ϕ is a drawing in the plane with the following properties, see Figure 1a):

- Variables in $\mathcal{U}$ and clauses in $\mathcal{C}$ are represented by pairwise disjoint rectangles in the plane, each of whose sides is parallel to the horizontal or the vertical axis.

- The horizontal axis intersects each rectangle representing a variable in $\mathcal{U}$ and no rectangle representing a clause in $\mathcal{C}$. Further, each rectangle representing a positive clause in $\mathcal{C}$ is drawn above the horizontal axis and each rectangle representing a negative clause in $\mathcal{C}$ is drawn below the horizontal axis.

- For each variable $x \in \mathcal{U}$ and for each clause $C \in \mathcal{C}$ such that C contains x or $\bar{x}$, there is a vertical line segment that joins the rectangles representing x and C and that intersects neither other line segments nor other rectangles.

Given a monotone rectilinear representation of a 3-SAT formula ϕ, the PLANAR MONOTONE 3-SAT PROBLEM is to decide whether ϕ is satisfiable. The following holds.

Theorem 2 (de Berg and Khosravi, Theorem 1 in [2])**.** *The* PLANAR MONOTONE 3-SAT PROBLEM *is NP-complete.*

Consider a planar monotone 3-SAT formula $\phi = \phi(\mathcal{U}, \mathcal{C})$ together with a monotone rectilinear representation $\mathcal{R}$. Throughout this paper, we do not distinguish between variables of ϕ and their representations as rectangles in $\mathcal{R}$, and similarly for clauses. Furthermore, without loss of generality, we assume that each variable in $\mathcal{U}$ appears in at least one positive clause and at least one negative clause, i.e., ϕ uses both literals x and $\bar{x}$ for each $x \in \mathcal{U}$. Also, we assume that each clause contains exactly three (not necessarily distinct) literals and, in $\mathcal{R}$, there are as many line segments joining x to C as x appears in C. Moreover, we assume that each variable x in $\mathcal{R}$ can be split into a left and a right part such that the line segments joining x to positive clauses touch x in its left part and line segments joining x to negative clauses touch x in its right part, see also Figure 1a).

2.2. Grids and Subdivisions of Grids

Consider two graphs H and G. Recall that H is a subdivision of G if H can be obtained from G by repeatedly subdividing edges. We say that two paths P and Q are *internally vertex-disjoint* if every vertex $v \in V(P)$ with $\deg_P(v) = 2$ satisfies $v \notin V(Q)$ and vice versa. Now, if H is isomorphic to a subdivision of G, then there are two injective maps

$$f_V : V(G) \to V(H) \quad \text{and} \quad f_E : E(G) \to \{P \subseteq H \colon\ P \text{ is a path in } H\}$$

such that, for all $\{u, v\} \in E(G)$, the path $f_E(\{u, v\})$ is an $f_V(u)$,$f_V(v)$-path that uses no vertex $f_V(w)$ with $w \in V(G) \setminus \{u, v\}$ as well as that the paths $f_E(e)$ and $f_E(e')$ are internally vertex-disjoint for all distinct $e, e' \in E(G)$. In the following, any such maps f_V and f_E are called *vertex and edge maps for H*.

Throughout this paper, let $[n] := \{1, \dots, n\}$ for $n \in \mathbb{N}$. For each $k \geq 3$, the $k \times k$ grid, denoted by $\tilde{G}_k$, is the graph with vertex set $\tilde{V}_k := \{(i, j) \colon\ i, j \in [k]\}$ and edge set

$$\tilde{E}_k := \{\{(i, j), (i', j')\} \colon\ |i - i'| + |j - j'| = 1\}\,.$$

Fix some $k \geq 3$ and let $\tilde{G} := \tilde{G}_k$. The *canonical embedding* of $\tilde{G}$ refers to a drawing of $\tilde{G}$ in the plane, where the vertex $(i, j) \in V(\tilde{G})$ is embedded at the point (i, j) in a coordinate system whose horizontal axis refers to the first coordinate and whose vertical axis refers to the second coordinate and each edge of $\tilde{G}$ is represented by a line segment. The unique infinite face of the canonical embedding of $\tilde{G}$ is called the *outer face* of $\tilde{G}$ and all other faces of $\tilde{G}$ are referred to as *inner faces* of $\tilde{G}$. For an inner face f of $\tilde{G}$, the terms *left/right/top/bottom edge* as well as the terms *face directly left/right/below/above* are defined in the obvious way according to the canonical embedding of $\tilde{G}$. For $i \in [k]$, the path that is induced in $\tilde{G}$ by the vertices in $\{(i, j) \colon j \in [k]\}$ is called the i^{th} *vertical path* of $\tilde{G}$ and, for $j \in [k]$, the path that is induced in $\tilde{G}$ by the vertices in $\{(i, j) \colon i \in [k]\}$ is called the j^{th} *horizontal path* of $\tilde{G}$. Edges of $\tilde{G}$ are called either *vertical* or *horizontal*, accordingly.

The notions defined above for grids naturally extend to subdivisions of grids, through vertex and edge maps. The *outer face* of H is the unique face of H whose boundary does not contain a vertex of degree 4. Let H be a graph isomorphic to a subdivision of $\tilde{G}$. For $i \in [k]$, the i^{th} *vertical path* P_i^v of H refers to the subdivision of the i^{th} vertical path of $\tilde{G}$. For $j \in [k]$, the j^{th} *horizontal path* P_j^h of H is defined analogously. Clearly, the paths $P_1^v, \ldots, P_k^v$ are pairwise vertex-disjoint and the paths $P_1^h, \ldots, P_k^h$ are also pairwise vertex-disjoint. The paths P_i^v, P_j^h with $i, j \in [k]$ are called *grid-paths* of H. By the *type* of a grid-path, we refer to the property of the grid-path being horizontal or vertical. For all $i, j \in [k]$, the unique common vertex of the paths P_i^v and P_j^h is called an *intersection vertex* of H.

Observation 3. *Let H be a subdivision of a $k \times k$ grid and let P and P' be two distinct grid-paths of H. Then $|V(P) \cap V(P')| \leq 1$ is satisfied. Furthermore, $|V(P) \cap V(P')| = 0$ (resp. $= 1$) if and only if P and P' are of the same (resp. different) type. In particular, P and P' do not have a common edge.*

A planar graph G *is uniquely embeddable* if all embeddings of G in the plane define the same set of facial boundaries, see [4] for details. Clearly, if a planar graph G is uniquely embeddable, then also any subdivision of G is uniquely embeddable. Consider a subdivision $\tilde{G}$ of the $k \times k$ grid with $k \geq 3$.

Due to Whitney's Theorem (see Theorem 4.3.2 in [4]) and the fact that $\tilde{G}$ is a subdivision of a 3-connected graph, $\tilde{G}$ is uniquely embeddable. This implies the following observation.

Observation 4. *Consider a plane graph G and a subgraph $H \subseteq G$ that is isomorphic to a subdivision of the $k \times k$ grid for some $k \geq 3$. Each vertical path P crosses each horizontal path Q at their unique common vertex v, i.e., a small circle around v touches P and Q alternately.*

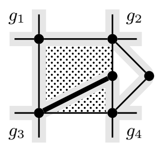
2.3. Normal Subdivisions, Bends, and Propagating Bends

For the remaining subsection, fix an integer $k \geq 3$ and let $\tilde{G}$ be the canonical embedding of the $k \times k$ grid. In the following section, we will only apply certain modifications to construct gadgets. By *basic modification*, we refer to the actions of deleting an edge, subdividing an edge, adding a new vertex, and adding a new edge. Consider a plane graph G obtained from $\tilde{G}$ by basic modifications. Clearly, G contains all vertices of $\tilde{G}$ and each such vertex is called a *grid-vertex*. For $X \subseteq V(\tilde{G})$, a graph $H \subseteq G$ that is isomorphic to a subdivision of $\tilde{G}$ is called *X-normal* if the outer face of $\tilde{G}$ is the outer face of H (i.e., their boundaries coincide) and a vertex map f_V for H can be chosen such that $f_V(w) = w$ for all $w \in V(\tilde{G}) \setminus X$. Every time we consider an X-normal subdivision, we implicitly fix such a choice of f_V. Observation 5 follows.

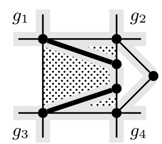
Observation 5. *Let G be a plane graph obtained from $\tilde{G}$ by basic modifications and let $H \subseteq G$ be an X-normal subdivision for some $X \subseteq V(\tilde{G})$.*

a) *Each vertex $(i, j) \in V(\tilde{G}) \setminus X$ is the unique common vertex of the i^{th} vertical path of H and the j^{th} horizontal path of H.*
b) *For each $i \in [k]$, if the i^{th} vertical path of H uses a grid-vertex $w \neq (i, j)$ for all $j \in [k]$, then $w \in X$. Similarly, for each $j \in [k]$, if the j^{th} horizontal path of H uses a grid-vertex $w \neq (i, j)$ for all $i \in [k]$, then $w \in X$.*

In order to construct the gadgets, we apply basic modifications to certain inner faces of $\tilde{G}$. These basic modifications often split an inner face of $\tilde{G}$ into



a) In H, $e = \{g_3, g_4\}$ bends into f.



b) In H, $e = \{g_1, g_3\}$ bends into f.

Figure 2: Examples for bends. The vertices g_i with $i \in [4]$ are grid-vertices. Let f be the face of $\tilde{G}$ whose area is dotted. The highlighted subgraph H is assumed to be X-normal with $g_i \notin X$ for $i \in [4]$.

several faces or change the boundary of an inner face. In this work, for a plane graph G obtained from $\tilde{G}$ through basic modifications, by abuse of notation, we refer to faces of G as the corresponding inner face of $\tilde{G}$ with the updated boundary if necessary. Further, for a face f of G, we say that a basic modification is *outside* f if no edge on the boundary of f is deleted and neither new edges nor new vertices are embedded inside f.

In Section 1.1, we informally used the concept of "paths that have to bend in order to have a subdivision of a grid". Next, we formalize this idea, which is a crucial concept throughout the paper. Let G be a plane graph obtained from $\tilde{G}$ by basic modifications. If an edge $e = \{x, y\} \in E(\tilde{G})$ is in G or it has been subdivided in order to obtain G, we use P_e to denote the x,y-path in G that replaces e and we assume that P_e is embedded exactly where e was embedded. Otherwise, if $e = \{x, y\} \in E(\tilde{G})$ has been deleted, we define $V(P_e) = \{x, y\}$ and $E(P_e) = \emptyset$. Consider an X-normal subdivision $H \subseteq G$ for some $X \subseteq V(\tilde{G})$ and denote by f_V a vertex map for H. For an inner face f of $\tilde{G}$ and the grid-path Q of H that contains $f_V(x)$ and $f_V(y)$ for some edge $e = \{x, y\}$ on the boundary of f, we say that the *path Q bends* into f, if the part of Q that joins $f_V(x)$ and $f_V(y)$ uses an edge $e' \notin E(P_e)$ that is embedded inside f or belongs to the boundary of f, see Figure 2. For the sake of simplicity, we also say that, in H, the *edge e bends* into f, even though e is not inside f and e might not be an edge of H. For $d \in \mathbb{N}$, an *f_0,f_d-face-sequence* is an alternating sequence $(f_0, e_1, f_1, \ldots, e_d, f_d)$ of distinct faces and edges of $\tilde{G}$ such that e_h is on the boundary of the faces f_{h-1} and f_h for all $h \in [d]$. Consider a face-sequence $F := (f_0, e_1, f_1, \ldots, e_d, f_d)$ with $d \geq 1$ and let e_0 be the edge on the boundary of f_0 with $e_0 \cap e_1 = \emptyset$. We say that H *bends* along F if e_h bends into f_h for all $h \in [d] \cup \{0\}$. Fix some $X \subseteq V(\tilde{G})$ and consider the set $\mathcal{H}$ of all X-normal subdivisions in G where e_0 bends into f_0. If $\mathcal{H} = \emptyset$ or every graph $H \in \mathcal{H}$ bends along F, then we say that *all X-normal subdivisions in G propagate bends along F.* Roughly speaking, this means that, if an X-normal subdivision has a bend in the first face of F, then there are bends in all faces of F. Observe that, $\mathcal{H} = \emptyset$ means that there is no X-normal subdivision in G where e_0 bends into f_0.

3. Gadgets

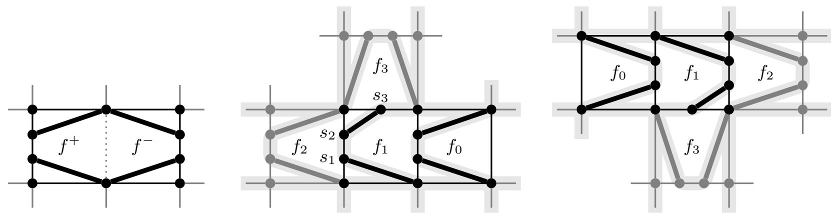
This section presents the gadgets, namely variable gadgets, clause gadgets, and wire gadgets. In Section 4, these gadgets are used to construct a graph G_ϕ that depends on a monotone rectilinear representation of a 3-SAT formula ϕ.

Throughout this section, we fix some large $k \in \mathbb{N}$ and denote by $\tilde{G}$ the canonical embedding of the $k \times k$ grid. When presenting the gadgets in the following subsections, we always assume that the gadgets are placed far enough from the outer face of $\tilde{G}$. In the figures showing gadgets, new edges are drawn thicker than edges in $\tilde{G}$ and edges arising from subdivisions. Vertices and edges that do not belong to the gadget subgraph are colored gray.

Before starting with the variable gadget, we quickly introduce *arrows*, which is a common concept used in the construction of the gadgets. Adding a *left arrow* to some inner face f of $\tilde{G}$, is defined by the following series of basic modifications. Let $i, j \in [k]$ be such that f is bounded by the cycle

$$((i, j), (i + 1, j), (i + 1, j + 1), (i, j + 1)).$$

First, replace the edge $\{(i, j), (i, j + 1)\}$ by the path $((i, j), s_1, s_2, (i, j + 1))$, where s_1 and s_2 are new vertices. Second, add the edges $\{(i + 1, j), s_1\}$ and



a) Decision gadget. **b)** Positive bifurcation. **c)** Negative bifurcation.

Figure 3: Decision and bifurcation gadgets. In Part b) and c), the highlighted subgraph visualizes, how the bend in f_0 is split into two bends in the connection faces f_2 and f_3.

$\{(i+1, j+1), s_2\}$. Similarly, we define a *right arrow*, an *up arrow*, and a *down arrow*.

3.1. Variable Gadgets

The variable gadget consists of one decision gadget and several bifurcation gadgets. The decision gadget is the part of the variable gadget that encodes the assignment of True or False to the variable. The bifurcation gadgets replicate the information encoded by the decision gadget as many times as needed, i.e., once for each line segment touching that variable in the monotone rectilinear representation.

Decision Gadget

Consider a vertical edge e of $\tilde{G}$ and denote by f^+ and f^- the two inner faces of $\tilde{G}$ whose right edge is e and whose left edge is e, respectively. To add a decision gadget for $x \in \mathcal{U}$ means to add a left arrow in f^+, to add a right arrow in f^-, and to delete the edge e, see Figure 3a). The face f^+ (resp. f^-) is called the *positive* (resp. *negative*) *face* of the variable x, and the new edges in f^+ (resp. f^-) are called the *positive* (resp. *negative*) *edges* of the variable x.

Bifurcation Gadget

Roughly speaking, a *positive bifurcation gadget* consists of two left arrows, where one of them is slightly twisted, see Figure 3b). More precisely, consider two inner faces f_0 and f_1 of $\tilde{G}$ such that f_1 is directly left of f_0. Denote by e_1 the unique edge of $\tilde{G}$ that is on the boundary of f_0 and f_1. Let e_2 and e_3 be the left edge and the top edge of f_1, respectively. The following basic modifications are applied to add a positive bifurcation gadget in f_0 and f_1. Add a left arrow in f_0. Subdivide e_2 twice, say with vertices s_1 and s_2, and subdivide e_3 once, say with the vertex s_3. Without loss of generality, assume that s_2 and s_3 have a common neighbor. Insert the new edge $\{s_2, s_3\}$ and a new edge joining s_1 to the common vertex of the bottom edge and the right edge of f_1. The faces f_1 and f_0 are called *bifurcation faces*, and f_0 is also called the *right connection* face of the positive bifurcation gadget. Moreover, the inner face f_2 of $\tilde{G}$, which is directly left of f_1, is called the *left connection face* and the inner face f_3 of $\tilde{G}$, which is directly above f_1, is called the *top connection face* of the positive bifurcation gadget. The faces f_2 and f_3 do not belong to the positive bifurcation gadget but are used to connect it to other gadgets as indicated in Figure 3b).

The *negative bifurcation gadget* is obtained by rotating the positive bifurcation gadget around 180 degrees, see Figure 3c) and the terminology for the negative bifurcation gadget is naturally adapted. Observe, that the negative bifurcation gadget defines a left, a right, and a bottom connection face but only the left connection face belongs to the gadget itself.

Assembling the Variable Gadget

In the following, we consider an instance of PLANAR MONOTONE 3-SAT, i.e., a monotone rectilinear representation $\mathcal{R}$ of a 3-SAT formula $\phi = \phi(\mathcal{U}, \mathcal{C})$. Fix a variable $x \in \mathcal{U}$. We denote by $\deg^+(x)$ (resp. $\deg^-(x)$) the number of appearances of x (resp. $\bar{x}$) in the clauses in $\mathcal{C}$, i.e., the number of line segments touching x in the top (resp. bottom) in $\mathcal{R}$. A *variable gadget* for x consists of $\deg^+(x)$ positive bifurcation gadgets and $\deg^-(x)$ negative bifurcation gadgets as well as two connection faces, see Figure 4. More precisely,

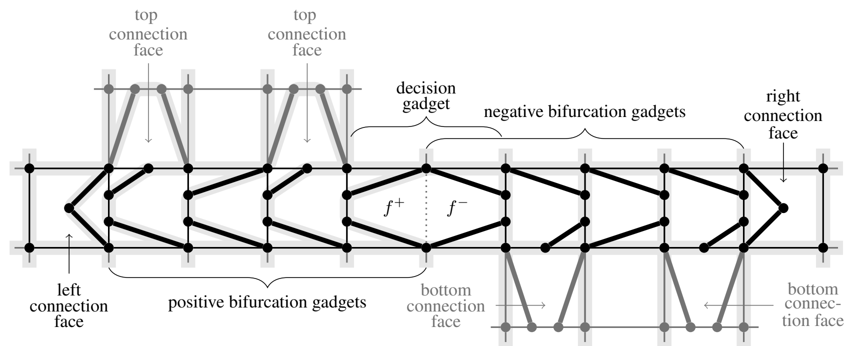


Figure 4: The variable gadget. In the highlighted subgraph, the edge e_x bends into the positive face f^+, which results in bends in all top connection faces.

let $(f_0, e_1, f_1, \ldots, e_{d+1}, f_{d+1})$ be a face-sequence with $d := 2(\deg^+(x) + \deg^-(x))$ such that f_h is the face directly right of f_{h-1} for all $h \in [d+1]$. The variable gadget consists of the faces $f_0, f_1, \ldots, f_{d+1}$ with the following basic modifications. A new vertex v_0, which is adjacent to both ends of e_1, is inserted in f_0. For each $h \in [\deg^+(x)]$, the faces f_{2h-1} and f_{2h} are modified according to the positive bifurcation gadget. The edge $e_{2\deg^+(x)+1}$ is deleted. For each h with $\deg^+(x) + 1 \leq h \leq \deg^+(x) + \deg^-(x)$, the faces f_{2h-1} and f_{2h} are modified according to the negative bifurcation gadget. A new vertex v_{d+1}, which is adjacent to both ends of e_{d+1}, is inserted in f_{d+1}. This completes the construction of the variable gadget, see Figure 4. Observe that the faces $f_{2\deg^+(x)}$ and $f_{2\deg^+(x)+1}$ are automatically modified according to the decision gadget, which we refer to as the decision gadget for x. The faces f_0 and f_{d+1} are now called the *left and the right connection face* of the variable gadget for x, respectively, and each top (resp. bottom) connection face of a positive (resp. negative) bifurcation gadget is now referred to as *top (resp. bottom) connection face* of the variable gadget. The variable gadget defines a left, a right, several top, and several bottom connection faces, but only the left and the right connection face belong to the gadget. The top and bottom connection faces will be used as origins for wires, which connect the variable gadget to clause gadgets.

Consider a graph G obtained from $\tilde{G}$ by adding one variable gadget corresponding to a variable $x \in \mathcal{U}$ and possibly further basic modifications outside the faces belonging to the variable gadget. Denote by f^+ and f^- the positive and the negative face of the decision gadget that is part of the variable gadget. Let $\hat{\mathcal{F}}_x^+$ be the family of face-sequences that, for each face f, where f is a top connection face or the left connection face, contains the f^+,f-face-sequence, that uses only positive bifurcation faces and the face f. Similarly, define $\hat{\mathcal{F}}_x^-$ to be a family of face-sequences starting in f^- and ending in a bottom connection face or the right connection face. Moreover, the unique edge of $\tilde{G}$ that was deleted due to the variable gadget for x, i.e., the unique edge on the boundary of f^+ and f^-, is denoted by e_x and called the *edge of x*.

Claim 6. *Let G be a plane graph obtained from $\tilde{G}$ by adding a variable gadget for $x \in \mathcal{U}$, an up arrow in each top connection face, and a down arrow in each bottom connection face as well as by further basic modifications outside the faces of the variable gadget and its connection faces. Consider a set $X \subseteq V(\tilde{G})$ containing no vertex of the variable gadget and its connection faces.*

a) *In every X-normal subdivision $H \subseteq G$, the edge e_x either bends into the positive face of x or into the negative face of x.*
b) *All X-normal subdivisions in G propagate bends along each face-sequence in $\hat{\mathcal{F}}_x^+$ and $\hat{\mathcal{F}}_x^-$.*

3.2. Clause Gadgets

Consider a positive clause $C \in \mathcal{C}$ and recall that, by assumption, C contains precisely three literals. A *positive clause gadget* for C consists of a *clause face* f, which is an inner face of $\tilde{G}$, and three *connection faces*, which are the faces directly left, below, and right of f. Figure 5 depicts the positive clause gadget. With reference to the vertex names in Figure 5, define $X_C = \{g_2, g_3, g_6, g_7\}$. The following claim states a crucial property of the positive clause gadget.

Claim 7. *Fix an inner face f of $\tilde{G}$ and denote by F the set of all inner faces of $\tilde{G}$ whose boundary contains a vertex on the boundary of f. Consider a plane*

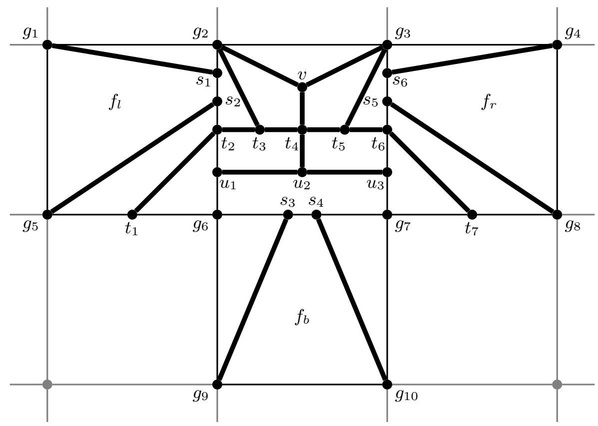


Figure 5: A positive clause gadget. The face of $\tilde{G}$ that is bounded by the cycle (g_6, g_7, g_3, g_2) is the clause face. The vertices g_h with $h \in [10]$ are grid-vertices.

graph G obtained from $\tilde{G}$ by adding a positive clause gadget with clause face f and further basic modifications outside the faces in F. Let $X \subseteq V(\tilde{G})$ be a set that contains X_C and such that $X \setminus X_C$ contains no vertex that is on the boundary of a face in F. The following holds.
There is no X-normal subdivision $H \subseteq G$ where e_h bends into f_h simultaneously for each connection face f_h of the clause gadget, where e_h denotes the unique edge of $\tilde{G}$ that is on the boundary of f_h and contains no vertex on the boundary of f.

Proof. Let G, f, F, and X be as in the statement. Denote by f_l, f_b, and f_r the faces directly left, below, and right of f, respectively. In the following, we use the vertex names as in Figure 5. Choose $i, j \in [k]$ such that $g_6 = (i, j)$.

Assume for a contradiction that there is an X-normal subdivision $H \subseteq G$ where $\{g_1, g_5\}$ bends into f_l, $\{g_9, g_{10}\}$ bends into f_b, and $\{g_4, g_8\}$ bends into f_r. Denote by $P_1^v, \ldots, P_k^v$ and $P_1^h, \ldots, P_k^h$ the vertical and horizontal paths of H. Let V^* be the set of vertices that are embedded in a face in F or on the boundary of a face in F. Due to Observation 5a), we have that, using only vertices from V^*,

- the path P_i^v joins $g_9 = (i, j-1)$ and $(i, j+2)$,
- the path P_{i+1}^v joins $g_{10} = (i+1, j-1)$ and $(i+1, j+2)$, and
- the path P_j^h joins $g_5 = (i-1, j)$ and $g_8 = (i+2, j)$.

The drawing of G contains a face f' bounded by the paths (g_1, s_1, s_2, g_5) and $P_{\{g_1,g_5\}}$. As $\{g_1, g_5\}$ bends into f_l in H, the path $P_{\{g_1,g_5\}}$ and the segment of the path P_{i-1}^v that joins g_1 to g_5 forms a cycle C, such that the face f' lies completely on one side of C. Since distinct vertical paths do not intersect, the paths P_i^v and P_{i+1}^v cannot use any vertex on the boundary of f'. In particular, P_i^v uses neither s_1 nor s_2. Similarly, it follows that P_j^h uses neither s_3 nor s_4, and that P_{i+1}^v uses neither s_5 nor s_6. As P_{i-1}^v intersects with P_j^h in $g_5 = (i-1, j)$, we have that P_{i-1}^v must use $\{g_5, s_2\}$ and P_j^h must use $\{g_5, t_1\}$. Similarly, P_{j-1}^h must use $\{g_9, s_3\}$ and P_i^v must use $\{g_9, g_6\}$. Since neither P_j^h nor P_i^v can use s_3 and P_j^h and P_i^v have no common edge, it follows that P_j^h and P_i^v cannot intersect in g_6. Thus, P_j^h must use $\{t_1, t_2\}$. A similar argument shows that P_i^v must use the edges $\{g_6, u_1\}$ and $\{u_1, u_2\}$. Due to symmetry, it follows that P_{i+1}^v must use the edge $\{u_3, u_2\}$. This is a contradiction as P_i^v and P_{i+1}^v cannot both use u_2 due to Observation 3. □

The *negative clause gadget* is obtained by rotating the positive clause gadget around 180 degrees. For a negative clause C, the set X_C consists of all grid-vertices that are on the boundary of the clause face. Claim 7 holds analogously for the negative clause gadget.

3.3. Wire Gadgets

The purpose of the wire gadgets is to propagate the information from the variable gadgets to the clause gadgets. Let $F = (f_0, e_1, f_1, \dots, e_d, f_d)$ be a face-sequence for some integer $d \geq 0$. Then, F is called *straight* if $d \leq 1$ or, $e_h \cap e_{h+1} = \emptyset$ for all $h \in [d-1]$. Moreover, for an integer $h \in [d-1]$, the face-sequence F *is almost straight (and it turns at f_h)* if F is not straight but $F' := (f_0, e_1, \dots, f_h)$ and $F'' := (f_h, e_{h+1}, \dots, f_d)$ are straight.

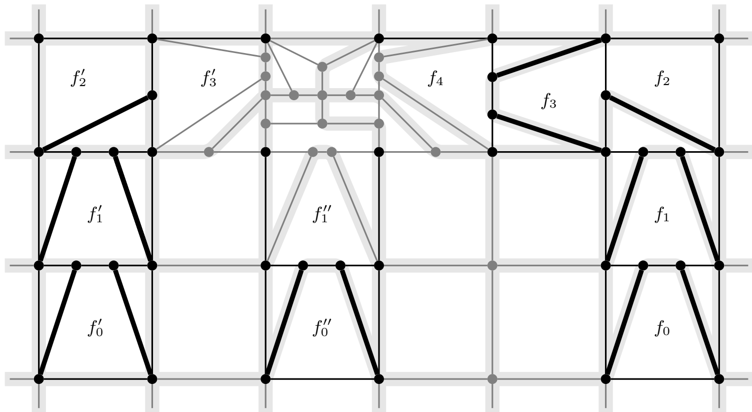


Figure 6: Wire gadgets along the face-sequences $F' = (f_0', e_1', f_1', e_2', f_2', e_3', f_3')$, $F'' = (f_0'', e_1'', f_1'')$, and $F = (f_0, e_1, f_1, e_2, f_2, e_3, f_3, e_4, f_4)$. The faces f_3', f_1'', and f_4 are connection faces of the clause gadget, which causes the modifications in these faces. The highlighted subdivision bends along F'' and F.

Now, fix some integer $d \geq 1$ and a straight or almost straight face-sequence $F = (f_0, e_1, f_1, \ldots, e_d, f_d)$. To add a *wire gadget* along F means to add an arrow to the faces $f_0, f_1, \ldots, f_{d-1}$ in such a way that the arrows point along F, see Figure 6. If F turns at f_h (even if $h = d - 1$), then the arrow added to f_h is modified to consist of only one edge, as the modifications in f_2' and f_2 in Figure 6 show. Observe that the last face of F is not modified. Later on, in Section 4, we will use wires, whose last face f_d is a connection face of a clause, which means that the clause gadget will add an arrow in f_d and possibly modify f_d further depending on whether f_d is a left, right, bottom, or top connection face of the clause.

The next claim follows from the structure of wire gadgets and Observation 5.

Claim 8. *Let $d \geq 1$ and let $F = (f_0, e_1, \ldots, e_d, f_d)$ be a straight or almost straight face-sequence in $\tilde{G}$. Consider a plane graph G that is obtained from $\tilde{G}$ by adding a wire gadget along F and further basic modifications outside the faces $f_0, \ldots, f_{d-1}$. Let $X \subseteq V(\tilde{G})$ be a set that contains no vertex of the* *faces $f_0, \dots, f_{d-1}$. Then, all X-normal subdivisions in G propagate bends along F.*

4. Construction of G_ϕ

Throughout this section and Section 5, fix an instance of the PLANAR MONOTONE 3-SAT PROBLEM, i.e., a monotone rectilinear representation $\mathcal{R}$ of a 3-SAT formula $\phi = \phi(\mathcal{U}, \mathcal{C})$. Define $n := |\mathcal{U}|$ and $m := |\mathcal{C}|$. Set $k := 8m+2n+5$. Denote by $\tilde{G}$ the $k \times k$ grid and consider $\tilde{G}$ together with its canonical embedding. In this section, we describe how to add gadgets to $\tilde{G}$ in order to obtain G_ϕ.

First, the variable gadgets are added one after another in the middle row of $\tilde{G}$. More precisely, let $F_\mathcal{U} := (f_1, e_2, f_2, \dots, f_{k-1})$ be the straight face-sequence, where f_1 is the inner face of $\tilde{G}$ whose boundary contains the grid-vertices $(1, \lfloor k/2 \rfloor)$ and $(1, \lceil k/2 \rceil)$ and f_{k-1} is the inner face of $\tilde{G}$ whose boundary contains the grid-vertices $(k, \lfloor k/2 \rfloor)$ and $(k, \lceil k/2 \rceil)$. Let $\mathcal{U} = \{x_1, \dots, x_n\}$ and assume that $x_1, \dots, x_n$ is the order in which the variables appear in the drawing $\mathcal{R}$. The variable gadgets for $x_1, \dots, x_n$ are placed, one after another, along $F_\mathcal{U}$ so that the left connection face of the variable gadget for x_1 is f_{m+3} and the right connection face of the variable gadget for x_n is $f_{7m+2n+2}$. In order to see that this is possible, set $d_i = 2(\deg^+(x_i) + \deg^-(x_i))$ for $i \in [n]$ and note that $\sum_{i \in [n]} \frac{1}{2} d_i$ counts each line segment of $\mathcal{R}$ once. Since, in $\mathcal{R}$, each clause touches exactly three line segments, we have $\sum_{i \in [n]} \frac{1}{2} d_i = 3m$ and precisely $(d_1 + 2) + \ldots + (d_n + 2) = 6m + 2n$ faces are occupied by all variable gadgets together.

Let B_{outer} be the set of vertices on the boundary of the outer face of $\tilde{G}$. For a face f of $\tilde{G}$, the *boundary distance* of f is defined as the length of a shortest v,w-path in $\tilde{G}$ such that v is on the boundary of f and $w \in B_{\text{outer}}$. Observe that every face of $\tilde{G}$ that belongs to a variable gadget has boundary distance at least $m + 2$. Denote by $G_\mathcal{U}$ the graph obtained from $\tilde{G}$ by adding the variable gadgets in the described way. In the following, more faces of $\tilde{G}$ in $G_\mathcal{U}$ are chosen for placing clause and wire gadgets. We choose one inner face f_C of $\tilde{G}$ for each clause $C \in \mathcal{C}$, which will be the clause face of the clause gadget for C, and a straight or almost straight face-sequence $\hat{F}_L$ in $\tilde{G}$ for each line segment L in $\mathcal{R}$ such that the following properties hold (see Figure 1 in Section 1.1 for an example).

(P1) The faces f_C with $C \in \mathcal{C}$ are pairwise distinct and, for all line segments L in $\mathcal{R}$ and all clauses $C \in \mathcal{C}$, the face-sequence $\hat{F}_L$ does not use f_C.

(P2) For each line segment L in $\mathcal{R}$ that joins a variable $x \in \mathcal{U}$ to a positive (resp. negative) clause $C \in \mathcal{C}$, the face-sequence $\hat{F}_L = (f_0, e_1, f_1, \ldots, f_d)$ satisfies: $d \geq 1$, face f_0 is a top (resp. bottom) connection face of the variable gadget corresponding to x, face f_1 is the face directly above (resp. below) f_0, and face f_d is a connection face of the clause C, i.e., the face directly right, below (resp. above), or left of f_C.

(P3) For all faces f_1 and f_2 of $\tilde{G}$ such that f_2 is the face directly above or right of f_1, the following holds. If f_1 and f_2 belong to the face-sequences $\hat{F}_{L_1}$ and $\hat{F}_{L_2}$, respectively, then $L_1 = L_2$. (This ensures that distinct face-sequences are face-disjoint and not directly next to each other.)

(P4) For all line segments L in $\mathcal{R}$, each face in $\hat{F}_L = (f_0, e_1, f_1, \ldots, f_d)$ has boundary distance at least $m + 2$.

Due to the properties of the rectilinear drawing $\mathcal{R}$ and the definition of k, it is easy to check that we can make such choices. Now, starting with $G_{\mathcal{U}}$, for each positive (resp. negative) clause $C \in \mathcal{C}$, add a positive (resp. negative) clause gadget such that its clause face is f_C. Moreover, for each line segment L in $\mathcal{R}$, add a wire gadget along $\hat{F}_L$. Denote by G_ϕ the graph obtained in this way and define $X_\phi = \bigcup_{C \in \mathcal{C}} X_C$. The next observation is easy to verify.

Observation 9. *The graph G_ϕ is plane and has size polynomial in the size of ϕ. Moreover, all faces of $\tilde{G}$ that were modified in the construction of G_ϕ have boundary distance at least $m + 2$.*

To state a key property of G_ϕ, we extend the face-sequences $\hat{F}_L$. Consider a line segment L in $\mathcal{R}$ that joins a variable $x \in \mathcal{U}$ to a positive clause $C \in \mathcal{C}$

and denote by f the positive connection face of the variable gadget of x, which is the first face of $\hat{F}_L$. Recall the set $\hat{\mathcal{F}}_x^+$ defined at the end of Section 3.1 and note that $\hat{\mathcal{F}}_x^+$ contains exactly one face-sequence F that ends in f. Let F_L be the face-sequence obtained by glueing F and $\hat{F}_L$ together. Moreover, for each variable $x \in \mathcal{U}$, denote by $\mathcal{F}_x^+$ the set of face-sequences that contains F_L for each line segment L in $\mathcal{R}$ that joins x to a positive clause in $\mathcal{C}$. Similarly, define F_L for line segments L in $\mathcal{R}$ that join a variable to a negative clause as well as $\mathcal{F}_x^-$ for negative occurrences of x. The following claim holds due to Claim 6 and Claim 8.

Claim 10. *For each variable $x \in \mathcal{U}$, all X_ϕ-normal subdivisions in G_ϕ propagate bends along each face-sequence in $\mathcal{F}_x^+$ and $\mathcal{F}_x^-$.*

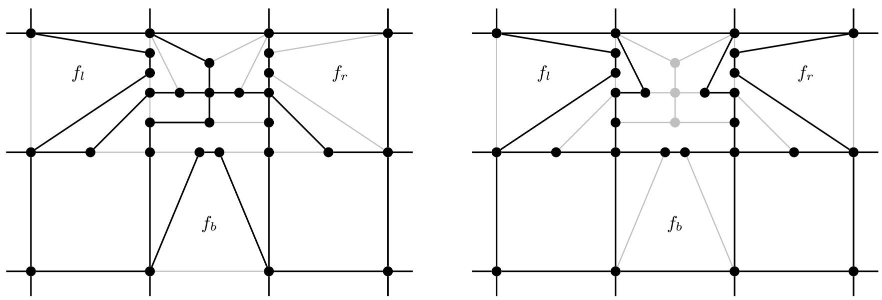
5. Reduction

Here, we show that the graph G_ϕ constructed in Section 4 contains a subdivision of a $k \times k$ grid if and only if the formula ϕ is satisfiable. To do so, we use the notation introduced in the previous sections.

5.1. If ϕ is Satisfiable

First, assume that ϕ is satisfiable. We prove that G_ϕ contains a subgraph H, which is isomorphic to a subdivision of the $k \times k$ grid. Fix a satisfying assignment $T : \mathcal{U} \to \{\textsc{True}, \textsc{False}\}$ of ϕ. Recall that e_x is the unique edge of $\tilde{G}$ that was deleted due to the decision part of the variable gadget of x and define $E_\mathcal{U} = \{e_x \colon x \in \mathcal{U}\}$. Moreover, recall that, for each $e = \{x, y\} \in E(\tilde{G}) \setminus E_\mathcal{U}$, the path P_e denotes the x,y-path of G_ϕ that replaces e in the construction of G_ϕ. Let $\tilde{G}_s$ be the graph obtained from $\tilde{G}$ by replacing each edge $e \in E(\tilde{G}) \setminus E_\mathcal{U}$ by P_e. Clearly, $\tilde{G}_s$ is a subdivision of $\tilde{G}$ and it is a subgraph of $G_\phi + E_\mathcal{U}$. We argue now that it is possible to remove each edge $e \in E_\mathcal{U}$ from $\tilde{G}_s$ while maintaining a subdivision of a $k \times k$ grid by introducing some bends.

Let H be the subgraph of G_ϕ obtained from $\tilde{G}_s$ by the following modifications. For each variable $x \in \mathcal{U}$ with $T(x) = \textsc{False}$, the edge e_x is replaced



a) The edges e_l and e_b bend into f_l and f_b, respectively.

b) The edges e_l and e_r bend into f_l and f_r, respectively.

Figure 7: X_C-normal subdivisions in a positive clause gadget.

by the positive edges of x and the edge $\tilde{e}$ joining them in G_ϕ. In other words, instead of using e_x, in H, the edge e_x bends into the positive face of the variable gadget for x. Next, the grid-path of $\tilde{G}_s$ that uses the edge $\tilde{e}$ is repaired by bending the left edge of the positive face into the face on its left side. We repeat the repairing process sequentially until all repaired grid-paths are vertex-disjoint paths of G_ϕ, as indicated by the subgraphs in Figure 4 and Figure 6. The final graph H bends along each face-sequence in $\mathcal{F}_x^+$ with $T(x) = \textsc{False}$. Similarly, for each variable $x \in \mathcal{U}$ with $T(x) = \textsc{True}$, bend e_x into the negative face of x and repair the grid-paths so that H bends along each face-sequence in $\mathcal{F}_x^-$ with $T(x) = \textsc{True}$.

To see that the construction of H is feasible, it suffices to check that, in H, the edges on the boundary of the clause faces can bend into the clause faces. Consider a positive clause $C \in \mathcal{C}$; the following argument is easy to adjust for a negative clause. As in Section 3.2, denote by f_l, f_b, and f_r the connection faces of the clause C and, for each $h \in \{l, b, r\}$, let e_h be the unique edge on the boundary of f_h that contains no vertex on the boundary of the clause face f_C. Due to the examples in Figure 7 and symmetry, it is easy to see that there are X_ϕ-normal subdivisions where up to two edges e_h with $h \in \{l, b, r\}$ bend into f_h

simultaneously. Since T is a satisfying assignment, there is a variable x that is used by C and satisfies $T(x) = \textsc{True}$. Denote by L a line segment of $\mathcal{R}$ that joins x to C. Since the edge e_x bends into the negative face of x in H, the construction of H does not require any modifications along F_L. Consequently, there is an $h \in \{l, b, r\}$ such that e_h does not bend into f_h in H and the construction of H is feasible.

5.2. If G_ϕ Contains a Subdivision of a $k \times k$ Grid

Now, assume that G_ϕ contains a subdivision of a $k\times k$ grid. Here, we show that ϕ is satisfiable. The main part of the proof is the next lemma.

Lemma 11. *If G_ϕ contains a subgraph H that is isomorphic to a subdivision of a $k\times k$ grid, then H is X_ϕ-normal.*

Before proving the previous lemma, we use it to argue that ϕ is satisfiable. Assume that G_ϕ contains a subgraph H that is isomorphic to a subdivision of the $k\times k$ grid. Due to Lemma 11, H must be X_ϕ-normal. In order to define a truth assignment $T : \mathcal{U} \rightarrow \{\textsc{True}, \textsc{False}\}$ for ϕ, consider a variable $x \in \mathcal{U}$. According to Claim 6a), in H, the edge e_x of x, bends into the positive or into the negative face of x. Set $T(x) = \textsc{True}$ if and only if e_x bends into the negative face of x. For a contradiction, assume that there is a positive clause $C \in \mathcal{C}$ that is not satisfied by the assignment T; the following is easy to adjust for a negative clause. Let x be an arbitrary variable used in C and denote by L a line segment in $\mathcal{R}$ that joins x to C. Then $T(x) = \textsc{False}$ and, hence, e_x bends into the positive face of x. By Claim 10, all X_ϕ-normal subdivisions in G_ϕ propagate bends along each face-sequence in $\mathcal{F}_x^+$ and therefore H bends along F_L. As, in $\mathcal{R}$, there are exactly three line segments that touch the clause C, there are three distinct face-sequences F_L ending in a connection face of C and H bends along each of these face-sequences. This contradicts Claim 7 and, hence, C must be satisfied. Consequently, ϕ has a satisfying assignment.

Throughout the remaining section, assume that G_ϕ contains a subgraph H, which is isomorphic to a subdivision of a $k\times k$ grid. We give a summary of the proof of Lemma 11 before presenting the details.

Sketch of Proof of Lemma 11

To prove Lemma 11, we have to show that

(i) the boundary of the outer face of $\tilde{G}$ is exactly the boundary of the outer face of H and

(ii) there is a vertex map f_V for H with $f_V(w) = w$ for all $w \in V(\tilde{G}) \setminus X_\phi$.

First, note that few new vertices of degree at least 4 are created when constructing G_ϕ from $\tilde{G}$ and these vertices are in the clause faces. Further, each clause face can contain at most 5 vertices with degree 4 in H (Claim 12). Thus, most of the intersection vertices of H with degree 4 must be grid-vertices. Due to Observation 9 and the aforementioned fact, there must be vertices of the outer face of H, which are vertices of the outer face of $\tilde{G}$ (Claim 14), and no vertex of the outer face of H belongs to a clause face (Claim 15). The previous fact allows us to prove that each clause face in fact contains at most 4 vertices with degree 4 in H (Claim 16). We conclude that the outer face of H coincides with the outer face of G_ϕ, i.e., (i) is satisfied. Finally, a separation argument implies that (ii) is satisfied, which completes the proof that H is X_ϕ-normal.

Proof of Lemma 11

Throughout the proof of Lemma 11, whenever a gadget for some positive clause is considered, we use the vertex names introduced in Figure 5. Recall that $X_C = \{g_2, g_3, g_6, g_7\}$ for positive clauses $C \in \mathcal{C}$ and $X_\phi = \bigcup_{C \in \mathcal{C}} X_C$. Let $V_4(C) := \{g_2, g_3, g_6, g_7, t_2, t_4, t_6\}$, which is the set of vertices of degree at least 4 in G_ϕ that belong to the clause face of the clause gadget corresponding to C. Define $V_4(C)$ analogously for negative clauses $C \in \mathcal{C}$. For a graph G and $d \in \mathbb{N}$, let $V_d(G) := \{v \in V(G)\colon\ \deg_G(v) \geq d\}$. By construction,

$$|V_4(H)| = (k-2)^2, \qquad |V_3(H) \setminus V_4(H)| = 4(k-2), \quad \text{and} \tag{1}$$

$$|V_4(G_\phi)| = (k-2)^2 + 3m, \tag{2}$$

since each clause gadget contains 3 vertices v with $\deg_{G_\phi}(v) \geq 4$ that are not grid-vertices, namely t_2, t_4, and t_6.

Claim 12. *For every clause $C \in \mathcal{C}$ we have $|V_4(C) \cap V_4(H)| \leq 5$.*

Proof. Consider a positive clause $C \in \mathcal{C}$; the following argument is easy to adjust for a negative clause. Recall that the notation refers to Figure 5. First, note that it suffices to show that

$$\{g_6, t_2\} \nsubseteq V_4(H) \qquad \text{and} \qquad \{g_7, t_6\} \nsubseteq V_4(H). \tag{3}$$

Due to symmetry of the clause gadget, it suffices to show that $\{g_6, t_2\} \nsubseteq V_4(H)$.

Towards a contradiction, assume that $\deg_H(g_6) = \deg_H(t_2) = 4$. Denote by P_1 and P_1' the grid-paths of H that intersect in t_2. Due to Observation 4, we may assume that P_1 contains the subpath (s_2, t_2, u_1) and that P_1' contains the subpath (t_1, t_2, t_3). Similarly, let P_2 and P_2' be the grid-paths of H that intersect in g_6 and assume that $(u_1, g_6, g_9) \subseteq P_2$ and $(t_1, g_6, s_3) \subseteq P_2'$.

Case 1: P_1 and P_2 are of the same type. Since P_1 and P_2 both use u_1, we have $P_1 = P_2$ as otherwise Observation 3 was violated. Therefore, P_1' and P_2' are of the same type. As P_1' and P_2' both use t_1, again Observation 3 implies that $P_1' = P_2'$. But then, the paths P_1 and P_1' have two vertices in common, namely g_6 and t_2, which contradicts Observation 3.

Case 2: P_1 and P_2 are of distinct types. Then, u_1 is an intersection vertex of H with $\deg_H(u_1) \leq 3$. Moreover, u_1 is adjacent to the intersection vertices g_6 and t_2, which satisfy $\deg_H(g_6) = \deg_H(t_2) = 4$. This is a contradiction, as any vertex of degree at most 3 in a grid is adjacent to at most one vertex of degree 4. □

Recall that B_{outer} is the set of vertices on the boundary of the outer face of $\tilde{G}$. Denote by B_H the set of all vertices of H that belong to the boundary of the outer face of H. For each clause $C \in \mathcal{C}$, let $V(C)$ be the set of vertices that belong to the clause face f_C: namely, vertices that are on the boundary of f_C, embedded inside f_C or inserted to subdivide an edge on the boundary of f_C.

Throughout this section, define $W := \bigcup_{C \in \mathcal{C}} V_4(C)$ and $\overline{W} := V_4(G_\phi) \setminus W$. The next observation follows immediately from Claim 12 and (1).

Observation 13. *We have* $|V_4(H) \cap W| \leq 5m$ *and* $|V_4(H) \cap \overline{W}| \geq (k-2)^2 - 5m$.

Claim 14. *The set* B_H *contains at least one vertex from* B_{outer}.

Proof. Suppose that $B_H \cap B_{\text{outer}} = \emptyset$. Let G' be the graph obtained from G_ϕ by deleting all vertices in $V_4(G_\phi)$ and all vertices in B_{outer}. It is easy to see that each component of G' contains at most 3 vertices. So, at least every fourth vertex of B_H belongs to $V_4(G_\phi)$. Since the boundary of the outer face of a $k \times k$ grid contains $4(k-1)$ vertices, B_H contains at least $4(k-1)$ vertices. Hence, we have that $|B_H \cap V_4(G_\phi)| \geq k-1$. Now, (1) and (2) imply that

$$(k-2)^2 \stackrel{(1)}{=} |V_4(H)| \leq |V_4(G_\phi)| - |B_H \cap V_4(G_\phi)| \stackrel{(2)}{\leq} (k-2)^2 + 3m - (k-1),$$

which is equivalent to $k - 1 \leq 3m$ and contradicts the definition of k. □

Claim 15. *For each* $C \in \mathcal{C}$, *we have* $B_H \cap V(C) = \emptyset$.

Proof. We first prove that

$$\left|B_H \cap \overline{W}\right| \leq m. \tag{4}$$

Towards a contradiction, assume that B_H contains more than m vertices from $\overline{W}$. The vertices in $\overline{W}$ are grid-vertices, that have degree 4 in G_ϕ and that do not belong to a clause gadget. Since $|\overline{W}| = (k-2)^2 - 4m$, we have

$$|V_4(H) \cap \overline{W}| \leq |\overline{W}| - |B_H \cap \overline{W}| < (k-2)^2 - 5m,$$

which contradicts Observation 13. Thus, (4) is indeed satisfied.

For a contradiction, assume that there is a clause $C \in \mathcal{C}$ with $B_H \cap V(C) \neq \emptyset$. Claim 14 implies that there is a path $P \subseteq G_\phi$ with $V(P) \subseteq B_H$ that starts in a vertex $v \in V(C)$ and ends in a vertex $w \in B_{\text{outer}}$. Recall Property (P4), which implies that every face of $\tilde{G}$ that was modified when constructing G_ϕ has boundary distance at least $m+2$. Hence, f_C has boundary distance at least $m+2$ and P contains at least $m+2$ grid-vertices of G_ϕ that are not in W

and at least $m+1$ of these vertices have degree 4 in G_ϕ. Therefore, B_H contains at least $m+1$ vertices from $\overline{W}$, which contradicts (4). $\square$

Claim 16. *For each clause $C \in \mathcal{C}$, we have $|V_4(C) \cap V_4(H)| \leq 4$.*

Proof. Without loss of generality, consider a positive clause $C \in \mathcal{C}$. The following properties are used in the remaining proof without further mentioning them:

- Any two distinct grid-paths of H of the same type do not have a common vertex and any two grid-paths of H of different types have precisely one common vertex (Observation 3).
- Grid-paths of H cannot end in a vertex of $V(C)$, i.e., all intersection vertices of H that lie in $V(C)$ have degree 4 with respect to H (Claim 15).
- In the drawing of G, each horizontal path crosses each vertical path of H at their unique common vertex (Observation 4).

In order to prove the claim, we check that there are no 5 vertices in $V_4(C)$ that belong to $V_4(H)$ simultaneously. First, we show that

$$\{t_2, t_4, t_6\} \not\subseteq V_4(H). \tag{5}$$

Towards a contradiction, assume that $\deg_H(t_2) = \deg_H(t_4) = \deg_H(t_6) = 4$. Let P and P' be the grid-paths of H that intersect in t_4. Without loss of generality, we can assume that $(v, t_4, u_2) \subseteq P$ and $(t_3, t_4, t_5) \subseteq P'$. Analogously, there are grid-paths P_1, P_1', P_2, and P_2' of H with $(s_2, t_2, u_1) \subseteq P_1$, $(t_1, t_2, t_3) \subseteq P_1'$, $(s_5, t_6, u_3) \subseteq P_2$, and $(t_5, t_6, t_7) \subseteq P_2'$. Now, t_3 and t_5 are not intersection vertices of H and, hence, $P' = P_1' = P_2'$. Then, P, P_1, and P_2 are distinct grid-paths of the same type. Moreover, the path P cannot end in u_2 and, hence, it contains either the edge $\{u_2, u_1\}$ or the edge $\{u_2, u_3\}$. Therefore, P intersects P_1 or P_2, which is a contradiction.

Due to (3) and (5), it suffices to show the following three properties in order to finish the proof:

a) $\{g_2, g_3, g_7, t_2, t_4\} \nsubseteq V_4(H)$

b) $\{g_2, g_3, g_6, t_4, t_6\} \nsubseteq V_4(H)$

c) $\{g_2, g_3, g_6, g_7, t_4\} \nsubseteq V_4(H)$.

Recall that there are no modifications in the faces whose bottom edges are $\{g_1, g_2\}$, $\{g_2, g_3\}$, and $\{g_3, g_4\}$, respectively. We first prove a). For the sake of a contradiction, we assume that the degree of each vertex g_2, g_3, g_7, t_2, and t_4 in H is four. Then, there are grid-paths P, P', P_1, P_2 and P_2' of H with $(v, t_4, u_2) \subseteq P$, $(t_1, t_2, t_3, t_4, t_5) \subseteq P'$, $(s_2, t_2, u_1) \subseteq P_1$, $(u_3, g_7, g_{10}) \subseteq P_2$ and $(s_4, g_7, t_7) \subseteq P_2'$. Further, there are grid-paths Q_1 and Q_1' which intersect in g_2 and grid-paths Q_2, Q_2' which intersect in g_3.

Since $P_1 \neq P$ as well as that P_1 and P are of the same type, P_1 uses the edge $\{u_1, g_6\}$ and P uses the edge $\{u_2, u_3\}$. Thus, $P = P_2$. Moreover, P uses the edge $\{v, g_2\}$ or the edge $\{v, g_3\}$.

Case 1: P uses $\{v, g_3\}$. Since P and P' intersect in t_4, the path P' cannot use g_3 and hence uses the edge $\{t_5, t_6\}$. Moreover, P' and P_2' are distinct and of the same type as they both intersect P. Hence, P_2' uses the edge $\{t_7, g_8\}$ and $(t_6, s_5, s_6) \subseteq P'$. Since P' cannot use g_3, the path P' uses g_4 and we have $(t_5, t_6, s_5, s_6, g_4) \subseteq P'$. Further, $P \in \{Q_2, Q_2'\}$, say $P = Q_2$. The paths Q_2' and P' are distinct and of the same type. Therefore, the edges $\{g_3, g_4\}$, $\{g_3, s_6\}$, and $\{g_3, t_5\}$ are not contained in Q_2' and neither in $P = Q_2$. Consequently, these edges do not belong to H and $\deg_H(g_3) < 4$, which is a contradiction.

Case 2: P uses g_2. Then, $P \in \{Q_1, Q_1'\}$, say $P = Q_1$. Moreover, P' uses the edge $\{t_1, g_5\}$ since P_1 uses g_6 and intersects P' already in t_2. Then, $(s_2, s_1, g_1) \subseteq P_1$, since P_1 and P' already intersect in t_2 and since P and P_1 are distinct grid-paths of the same type, which implies that P_1 cannot use g_2. Now, Q_1' and P_1 are of different types. Hence, Q_1' and P_1 can only intersect in vertices with degree ≥ 4 in G_ϕ and thus Q_1' does not use s_1. Additionally, Q_1' cannot use t_3 because Q_1' and P' are of the same type. Analogously, it follows that P uses neither s_1 nor t_3. Thus, Q_1' uses the edge $\{g_2, g_3\}$ and $Q_1' \in \{Q_2, Q_2'\}$, say $Q_1' = Q_2'$. Now, P', P_2', and Q_1' are pairwise distinct grid-

paths of the same type. Therefore, $(g_7, t_7, g_8) \subseteq P'_2$ and $(t_5, t_6, s_5, s_6, g_4) \subseteq P'$. Since Q'_1 must use t_5, s_6, or g_4, the grid-paths Q'_1 and P' have a common vertex and are of the same type, which is a contradiction.
Statement a) is equivalent to b) due to symmetry of the clause gadget.

We now prove c). Towards a contradiction, assume that g_2, g_3, g_6, g_7, and t_4 are vertices of degree 4 in H. Then, there are grid-paths P, P', P_1, P'_1, P_2, and P'_2 of H with $(v, t_4, u_2) \subseteq P$, $(t_3, t_4, t_5) \subseteq P'$, $(u_1, g_6, g_9) \subseteq P_1$, $(t_1, g_6, s_3) \subseteq P'_1$, $(u_3, g_7, g_{10}) \subseteq P_2$, and $(s_4, g_7, t_7) \subseteq P'_2$. Now, $P'_1 = P'_2$ since P'_1 cannot use g_9 and P'_2 cannot use g_{10}. Moreover, P uses u_1 or u_3. Due to symmetry, we may assume that P uses u_1 and thus $P = P_1$. Let Q_1, Q'_1 and Q_2, Q'_2 be the grid-paths of H that intersect in g_2 and g_3, respectively. The path P uses either the edge $\{v, g_2\}$ or the edge $\{v, g_3\}$.

Case 1: P uses g_2. Then $P \in \{Q_1, Q'_1\}$, say $P = Q_1$. Similar to the arguments used in the proof of a), one can argue that $(t_3, t_2, s_2, s_1, g_1) \subseteq P'$ and $(t_1, g_5) \subseteq P'_1$. Since Q'_1 must use g_1, s_1, or t_3, the paths Q'_1 and P' have a common vertex and are of the same type, which is a contradiction.

Case 2: P uses g_3. Then, $(t_5, t_6) \subseteq P'$ and $(u_3, t_6) \subseteq P_2$. As P_2 and P' are of different types, they intersect in t_6. Consequently, $t_6 \in V_4(H)$, which contradicts (3). □

Claim 16 implies that $|V_4(H) \cap W| \leq 4m$, and $|\overline{W}| = (k-2)^2 - 4m$ holds by construction. Consequently, $\overline{W} \subseteq V_4(H)$ as otherwise (1) was violated.

Observation 17. *$V_4(G_\phi) \setminus W \subseteq V_4(H)$.*

Denote by $\tilde{P}^v_1, \ldots, \tilde{P}^v_k$ and $\tilde{P}^h_1, \ldots, \tilde{P}^h_k$ the vertical and horizontal paths of $\tilde{G}$. For each $i \in [k]$, define the set S^v_i as follows. If none of the vertices in $\tilde{P}^v_i$ is on the boundary of a face f_C with $C \in \mathcal{C}$, let $S^v_i = V(\tilde{P}^v_i)$. Otherwise, there is exactly one clause $C \in \mathcal{C}$ such that $\tilde{P}^v_i$ contains a vertex on the boundary of f_C due to the construction of G_ϕ. Assume that C is a positive clause. If $\{g_2, g_6\} \subseteq V(\tilde{P}^v_i)$, then define $S^v_i = (V(\tilde{P}^v_i) \setminus \{g_2, g_6\}) \cup \{g_1, g_5\}$, and otherwise let $S^v_i = (V(\tilde{P}^v_i) \setminus \{g_3, g_7\}) \cup \{g_4, g_8\}$. If C is a negative clause, define S^v_i

analogously. Moreover, for each $j \in [k]$, define $S_j^h = V(\tilde{P}_j^h)$. The next claim is easy to verify.

Claim 18. *For each $i \in [k]$, the set S_i^v separates $V(\tilde{P}_1^v)$ from $V(\tilde{P}_k^v)$ in G_ϕ and, for each $j \in [k]$, the set S_j^h separates $V(\tilde{P}_1^h)$ from $V(\tilde{P}_k^h)$ in G_ϕ.*

Now, we have all technical details to prove that H is X_ϕ-normal.

Proof of Lemma 11. Towards a contradiction, assume that $B_H \neq B_{\text{outer}}$. Due to Claim 14, the set B_H contains at least one vertex from B_{outer}. Hence, there is an edge $e = \{v, w\} \in E(H)$ with $v \in B_{\text{outer}}$ and $w \notin B_{\text{outer}}$. Then, $w \notin V_4(H)$ and due to Observation 9 we have $w \in V_4(G_\phi) \setminus W$, which together is a contradiction to Observation 17. Consequently, $B_H = B_{\text{outer}}$.

Let f_V be a vertex map for H. Since $B_H = B_{\text{outer}}$, we have

$$\{f_V((1,1)), f_V((1,k)), f_V((k,1)), f_V((k,k))\} \subseteq \{(1,1), (1,k), (k,1), (k,k)\}.$$

Due to symmetry of the $k \times k$ grid, we may assume without loss of generality that $f_V((i,j)) = (i,j)$ for all $(i,j) \in B_{\text{outer}}$. Denote by $P_1^v, \ldots, P_k^v$ and $P_1^h, \ldots, P_k^h$ the vertical and horizontal paths of H that correspond to the vertex map f_V. Next, we consider the grid-vertices of G_ϕ with degree ≥ 4 that do not belong to a clause gadget. Fix $(i,j) \in V_4(G_\phi) \setminus W$. Recall the sets S_j^h and S_i^v defined above and note that S_i^v contains (i,j). In H, there are exactly k horizontal paths. These paths are pairwise vertex-disjoint and each of them joins a vertex in $V(\tilde{P}_1^v)$ to a vertex in $V(\tilde{P}_k^v)$. Furthermore, the set S_i^v contains exactly k vertices. Hence, each horizontal path of H uses exactly one vertex from S_i^v, due to Claim 18. Since G_ϕ is plane and the horizontal paths do not intersect, the path P_j^h must use (i,j). Analogously, it follows that P_i^v must use (i,j). Consequently $f_V((i,j)) = (i,j)$ for all $(i,j) \in V_4(G_\phi) \setminus W$. Observing that $V(\tilde{G}) \setminus X_\phi \subseteq V_4(G_\phi) \setminus W$ implies that H is X_ϕ-normal.

References